Feb 21, 2017

Monday, February 20, 2017 11:30 PM

Previously, we have discussed this.

Uniqueness Theorem (Continuous extension)

Let X, Y be spaces where Y is Hausdorff;
$A \subset X$ and $\overline{A} = X$.
If $f, g : X \longrightarrow Y$ are continuous, $f|_A \equiv g|_A$
then $f \equiv g$ on X.

Note that not much condition on the spaces.

Existence Theorem Let $(X, d_X), (Y, d_Y)$ be metric spaces where Y is complete and $\overline{A} = X$.
If $f : A \longrightarrow Y$ is uniformly continuous
then $\exists$ unique continuous (in fact uniformly) extension $\tilde{f} : X \longrightarrow Y$, i.e., $\tilde{f}|_A \equiv f$.

Remarks.

1. Uniqueness comes from the previous theorem.
2. Both X, Y need metric because a "stronger" type of continuity is essential.
3. f is defined ~~on~~ A, no need to be X.

Definition. A mapping $f : (X, d_X) \longrightarrow (Y, d_Y)$ is uniformly continuous if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $\forall\, x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$ we have $d_Y(f(x_1), f(x_2)) < \varepsilon$.

02-21-p2

Monday, February 20, 2017 11:30 PM

Proof of Existence Theorem.

Let $x \in X$. The aim is to define $\tilde{f}(x)$.

Since $x \in X = \bar{A}$ and X is metric (1st countable), $\exists$ sequence $(a_n^x)_{n=1}^\infty$ in A, $a_n^x \longrightarrow x$ as $n \to \infty$.

In particular, $(a_n^x)_{n=1}^\infty$ is Cauchy.

Now, consider the image sequence $f(a_n^x)$ in Y

As Y is complete, we hope that $(f(a_n^x))_{n=1}^\infty$ is Cauchy and thus it converges.

In this case, we may define $\tilde{f}(x) = \lim f(a_n^x)$.

So now, we aim at showing $f(a_n^x)$ Cauchy, that is, for arbitrary $\varepsilon > 0$, we need $N \in \mathbb{Z}$ if $m, n \geq N$, $d_Y(f(a_m^x), f(a_n^x)) < \varepsilon$.

$$\Uparrow \text{ uniform continuity of } f$$

$d_X(a_m^x, a_n^x) < \delta$ for some $\delta > 0$

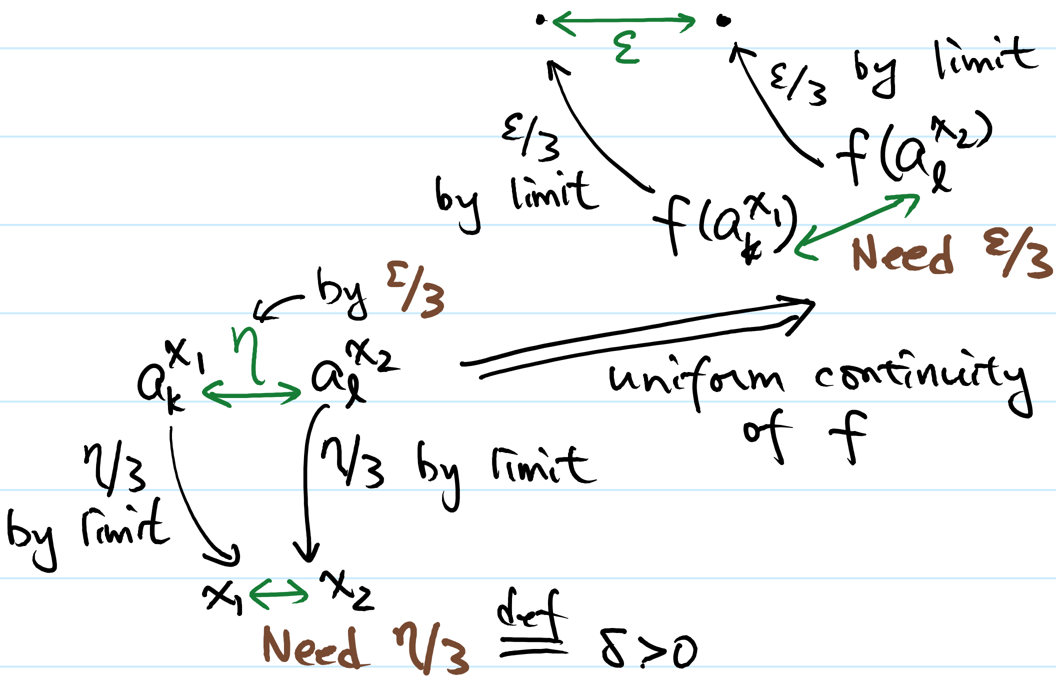
$$\Uparrow \ (a_n^x)_{n=1}^\infty \text{ Cauchy}$$

$N \in \mathbb{Z}$ can be chosen and $m, n \geq N$

Note that up to this point, even though $\tilde{f}(x)$ is defined, it may depend on the choice of sequence $a_n^x \longrightarrow x$.

This choice occurs at every $x \in X$.

Next, we want to prove $\tilde{f}$ is continuous.
Once this is done, we can apply the
Uniqueness Theorem and conclude that
there is only one $\tilde{f}$, so independent of
choice of sequences $(a_n^x)_{n=1}^{\infty}$.
We will show that $\tilde{f}$ is uniformly continuous.
So, we expect $d_Y(\tilde{f}(x_1), \tilde{f}(x_2)) < \varepsilon$



This diagram illustrates how we set up
the ε-δ-argument to prove the
uniform continuity of $\tilde{f}$.

02-21-p4

Wednesday, February 22, 2017 8:39 PM

Let us turn to another study. First, recall the logical statement for a dense set D in X.

$$X = \overline{D}$$

$$\forall x \quad \forall U \in \mathcal{J} \text{ with } x \in U, \quad U \cap D \neq \phi$$

$$\forall \phi \neq U \in \mathcal{J}$$

Now, let us consider the "opposite" of a dense set. Here, "opposite" is not negation.

Example. In $\mathbb{R}$, $\mathbb{Q}$ is dense as rational number is everywhere close to each other. One possible opposite is $\mathbb{Z}$, everyone is quite far from another integer.

Consider the topological condition on $\mathbb{Z}$.

* $\overline{\mathbb{Z}} = \mathbb{Z}$, this is the condition for closed
* $\mathring{\mathbb{Z}} = \phi$, but $\mathring{\mathbb{Q}} = \phi$, so not a characteristic
* $(\overline{\mathbb{Z}})^\circ = \phi$, not true for most subsets of $\mathbb{R}$

Definition. A subset $N \subset X$ is *nowhere dense* if $(\overline{N})^\circ = \phi$.

Examples.

(1) $\mathbb{Z}$ in $\mathbb{R}$.

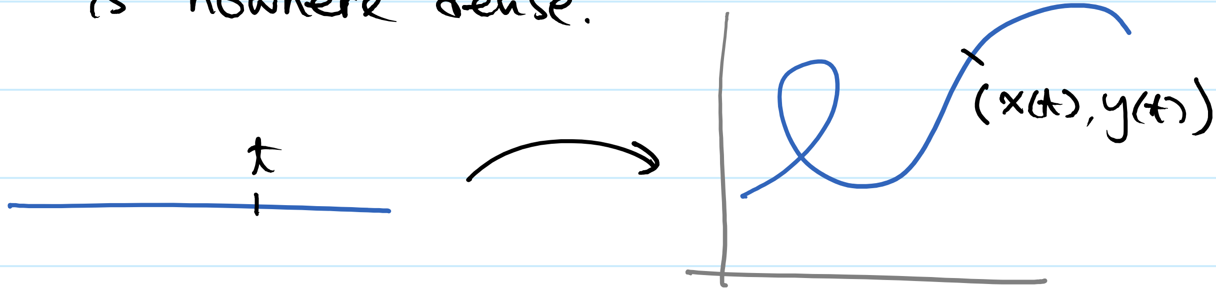
(2) $\mathbb{R}$ in $\mathbb{R}^2$, or $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$.

02-21-p5

Wednesday, February 22, 2017 8:50 PM

Example. Given "nice" functions $x(t), y(t)$

The set $\{(x(t), y(t)) : t \in \text{interval}\} \subset \mathbb{R}^2$ is nowhere dense.



Clearly, continuity is not "nice" enough because there is space-filling curve.

The natural "nice" condition is C^1 with $x'(t)^2 + y'(t)^2 \neq 0$ for all t.

Then one may use Inverse Function Theorem to show the set is nowhere dense.

Let us explore nowhere dense logically

$$(\bar{N})^\circ = \emptyset, \text{ i.e., } \forall\, x \in X \quad \underbrace{x \notin (\bar{N})^\circ}_{\forall\, U \in \mathcal{T} \text{ with } x \in U,\ U \not\subset \bar{N}}$$

Similar as before,

$$\forall\, \emptyset \neq U \in \mathcal{T}, \quad \underbrace{U \not\subset \bar{N}}_{U \cap (X \setminus \bar{N}) \neq \emptyset}$$

That means $X \setminus \bar{N}$ is dense.

Fact. N is nowhere dense $\Leftrightarrow$ $X \setminus \bar{N}$ is dense.

Note that both $\mathbb{R}\setminus\mathbb{Q}$ and $\mathbb{Q}$ are dense.
One may know that the Lebesgue measures of $\mathbb{R}\setminus\mathbb{Q}$ and $\mathbb{Q}$ are very different. But this is not topological.

Definition. A set $A \subset X$ is *of first category* if $A = \bigcup_{k=1}^{\infty} N_k$ where each N_k is nowhere dense. Otherwise, it is *of second category*.

Clearly, a countable union of 1st category sets is of 1st catagory.

Baire Category Theorem. Every complete metric space is of 2nd category.

Consequently, $\mathbb{R}$ is of 2nd category
As $\mathbb{R} = (\mathbb{R}\setminus\mathbb{Q}) \cup \mathbb{Q}$ and $\mathbb{Q}$ is 1st category,

$\mathbb{R}\setminus\mathbb{Q}$ must be of 2nd category.